MA3042 (4-0) Linear Algebra
Objectives

Following is an illustrative list of objectives for MA3042. It should be noted that the successful student will be able to synthesize solutions to problems not explicitly indicated in this list, and to prove simple theorems in the subject. It is assumed that the entering student has already mastered the elementary operations of matrix algebra by completing a preparatory course such as MA1042 or the equivalent. Also useful would be a course in the rudiments of logic and proof techniques, such as MA1025. The major headings in this list are taken from the corresponding chapter titles in *Linear Algebra with Applications*, 6th edition, Steven J. Leon, Prentice Hall 2002, but the objectives themselves are text-independent.

1. Vector Spaces

   Explain the defining properties of a vector space.

   Give examples of vector spaces.

   Given a set equipped with the necessary operations, explain why it is or is not a vector space.

   Given a subset of a given vector space V, explain why it is or is not a subspace of V.

   Find the nullspace of a given matrix.

   Explain what is meant by the span of a set of vectors.

   Given a vector space V and a collection B of vectors from V, explain why B is or is not a spanning set for V.

   Explain what is meant by linear independence of a collection of vectors.

   Given a collection S of vectors from a vector space V, explain why S is or is not a linearly independent set.

   Explain the defining properties of a basis for a vector space.

   Given a set S of vectors from a vector space V, explain why S is or is not a basis for V.

   Explain what is meant by the dimension of a vector space.

   Given a particular vector space V, find a basis for V.

   Given a vector $\mathbf{x}$ in a vector space V with basis U, find the coordinates of $\mathbf{x}$ with respect to a different basis W.

   Given a vector space V with bases U and W, find the transition matrix for change of basis from U to W and vice versa.

   Given an $m \times n$ matrix A, find bases for the nullspace, the row space, and the column space of A.

   Explain why the equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Given a consistent equation of the form $A\mathbf{x} = \mathbf{b}$, describe $\mathbf{b}$ as a linear combination of the columns of A.

Given an $m \times n$ matrix A, find the rank of A.

2. Linear Transformations

State the definition of a linear transformation L from a vector space V to another vector space W.

Give examples of linear transformations.

Given a mapping f from a vector space V to another vector space W, explain why f is or is not a linear transformation.

Given a linear transformation L from a vector space V to a vector space W, find the kernel of L.

Given a linear transformation L from a vector space V to a vector space W, and given a subspace S of V, find the image of S under L.

Given a linear transformation L from a vector space V to a vector space W, find the range of L.

Given a linear transformation L from $\mathbf{R}^n$ to $\mathbf{R}^m$, find the standard matrix representation of L.

Given a linear transformation L from a vector space V to a vector space W, find the matrix representation of L with respect to given bases E and F.

State the definition of similarity of square matrices.

Give examples of similar matrices.

Given a linear transformation L on a vector space V, and given two ordered bases E and F, find the matrix representations of L with respect to E and F and show that these two matrices are similar.

3. Orthogonality

Explain what is meant by the scalar product in $\mathbf{R}^n$.

Given two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$, find the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

Explain the concept of orthogonality of vectors.

Given vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$, determine whether they are orthogonal.

Given vectors $\mathbf{x}$ and $\mathbf{y}$ in a general vector space V, determine whether they are orthogonal.

Given a pair vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$, find the scalar and vector projections of $\mathbf{x}$ onto $\mathbf{y}$.

Given a vector $\mathbf{N}$ and a point p_0, find the equation of the plane normal to $\mathbf{N}$ that passes through p_0.

Explain the concept of orthogonality of subspaces.

Given a proper subspace S of a vector space V, find the orthogonal complement of S in V.

Explain the orthogonality relations among the four fundamental subspaces of a matrix.

Describe the relationship between the dimensions of the nullspace and the row space of a matrix. Do the same for the dimensions of the nullspace of the transpose and the column space.

Given a vector space, describe that space as a direct sum of two orthogonal subspaces.

Explain the following theorem: If A is an $m \times n$ matrix, and if $\mathbf{b} \in \mathbf{R}^m$, exactly one of the following is true:

(a) There exists $\mathbf{x} \in \mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.

(b) There exists $\mathbf{y} \in \mathbf{R}^m$ such that $A^T\mathbf{y} = 0$ but $\mathbf{y}^T\mathbf{b} \neq 0$.

Give definitions of inner product and inner product space.

Describe the fundamental properties of inner product spaces.

Given two orthogonal vectors from an inner product space, verify that they satisfy the Pythagorean Law.

Given vectors $\mathbf{u}$ and $\mathbf{v}$ from an inner product space V, find the scalar and vector projections of $\mathbf{u}$ onto $\mathbf{v}$.

State the Cauchy-Schwarz theorem.

State the properties of a vector norm.

Give examples of normed linear spaces.

Define the family of norms $||\cdot||_p$. Given a vector $\mathbf{v}$ in a normed linear space, find $||\mathbf{v}||_1,\ ||\mathbf{v}||_2$, and $||\mathbf{v}||_\infty$.

Given a least-squares problem, derive the normal equations and solve.

Explain what condition must be satisfied for the normal equations to have a unique solution.

Give the definition of an orthonormal set.

State the definition and fundamental properties of orthogonal matrices.

Give examples of orthogonal matrices.

Explain the advantage of working with orthogonal matrices in least-squares problems.

Be able to apply the Gram-Schmidt process or the modified Gram-Schmidt process to construct an orthonormal set of vectors in an inner product space.

Find the QR-factorization of a given matrix using the modified Gram-Schmidt process.

Apply the QR-factorization to the solution of least-squares problems.

4. Eigenvalues

State the definition of eigenvalue and eigenvector.

Given a square matrix, find its characteristic equation and solve to find the eigenvalues.

Given a square matrix and its eigenvalues, find associated eigenvectors.

Explain how the eigenvalues of similar matrices are related.

State necessary and sufficient conditions for a matrix to be diagonalizable.

Given a diagonalizable matrix A, construct a matrix X such that $X^{-1}AX = D$, a diagonal matrix.

Explain the application of diagonalization to computing the matrix exponential.

State the definitions of stochastic and doubly stochastic matrices.

State the definition of a Hermitian matrix, and give examples.

State the definition of a unitary matrix, and give examples.

State Schur's theorem.

State and apply the spectral theorem.

State the definition of a normal matrix.

Give necessary and sufficient conditions for a matrix to be normal, in terms of its eigenvectors.

Given an $m \times n$ matrix A, find the singular value decomposition of A.

Given the singular value decomposition $A = U\Sigma V^T$ of an $m \times n$ matrix A, explain the relationships among the columns of U and V and the four fundamental subspaces associated with A.

Given an $m \times n$ matrix A of rank r, and given $k < r$, use the singular value decomposition of A to find the matrix of rank k closest to A with respect to the Frobenius norm.

Given a quadratic equation in two variables, find the associated quadratic form.

Apply diagonalization to the simplification of quadratic forms.

Give definitions for positive (negative) definite (semidefinite) matrices.

Give necessary and sufficient conditions for real symmetric matrix to be positive (negative) definite (semidefinite).

Given a sufficiently differentiable function of two variables, use the Hessian to classify its stationary points.

State the properties of symmetric positive definite matrices.

Given a symmetric positive definite matrix A, factor A as $A = LDL^T$.

Given a symmetric positive definite matrix A, find its Cholesky decomposition.

Given a symmetric positive definite matrix A, factor A as $A = B^TB$.

5. Numerical Linear Algebra

Explain relative and absolute error in the floating-point representation of a number.

Given two floating-point numbers and a specified number of digits, perform prescribed arithmetic operations and calculate the resulting absolute and relative errors.

Given a matrix A, factor A as $PA = LU$, and use this factorization to solve equations of the form $A\mathbf{x} = \mathbf{b}$.

Describe the process of partial pivoting, and give examples.

Given a matrix A, find its Frobenius norm.

Describe the properties of matrix norms.

Explain what is meant by a subordinate matrix norm.

Define the condition number of a matrix.

Given an equation $A\mathbf{x} = \mathbf{b}$, briefly explain the relationship between the condition number of A, the relative error, and the relative residual.

Give the definition of an elementary orthogonal matrix.

Given a vector $\mathbf{x} \in \mathbf{R}^n$, construct a Householder matrix that will replace specified entries of $\mathbf{x}$ with zeroes.

Given a vector $\mathbf{x} \in \mathbf{R}^n$, construct a Givens transformation that will annihilate a specified entry of $\mathbf{x}$.

Use a sequence of orthogonal transformations to construct a QR-factorization of a matrix.